United Kingdom
Mathematics Trust

British Mathematical Olympiad

Round 2

Solutions

1. For a given positive integer k, we call an integer n a *k-number* if both of the following conditions are satisfied:

(i) The integer n is the product of two positive integers which differ by k.

(ii) The integer n is k less than a square number.

Find all k such that there are infinitely many k-numbers.

Solution

Note that n is a k-number if and only if the equation

$$n = m^2 - k = r(r + k)$$

has solutions in integers m, r with $k \geq 0$.

The right-hand equality can be rewritten as

$$k^2 - 4k = (2r + k)^2 - (2m)^2 ,$$

so k-numbers correspond to ways of writing $k^2 - 4k$ as a difference of two squares, $N^2 - M^2$ with $N > r$ and M even (which forces N to have the same parity as k).

Any non-zero integer can only be written as a difference of two squares in finitely many ways (because each gives a factorisation, and a number has only finitely many factors).

If $k \neq 4$ then $k^2 - 4k \neq 0$, and as a result, if $k \neq 4$ then there are only finitely many k-numbers.

Conversely, if $k = 4$ then setting $m = r + 2$ for $r \geq 0$ shows that there are infinitely many 4-numbers.

2. Find all functions f from the positive integers to the positive integers such that for all integers[a] x, y we have:

$$2yf(f(x^2)+x) = f(x+1)f(2xy).$$

[a]The final instance of the word *integers* was added retrospectively to avoid ambiguity.

SOLUTION

First substitute $x = 1$ to see that $f(2y) = ky$ for all positive integers y, where $k = \frac{2f(f(1)+1)}{f(2)}$. By taking $y = 1$, we get $f(2) = k$, so k is a positive integer.

Next, substitute $x = 2z$ and $y = 1$ to see that $f(2z+1) = kz+1$ for all positive integers z.

Then substitute $x = 2z+1$ and $y = 1$ to find that $k = 2$. So $f(x) = x$ for all integers $x \geq 2$.

Using $k = \frac{2f(f(1)+1)}{f(2)}$ we find that $f(1) = 1$, and so f is the identity.

This is easily checked to satisfy the functional equation.

REMARK

The question was originally posed without the final instance of the word *integers*. This gave rise to an alternative interpretation of the problem where x and y can be any numbers such that x^2, $f(x^2)+x, x+1$ and $2xy$ are integers. In this case x must be a positive integer, but y can be a positive half integer. This variant can be solved in a similar, though slightly quicker, way.

3. The cards from n identical decks of cards are put into boxes. Each deck contains 50 cards, labelled from 1 to 50. Each box can contain at most 2022 cards. A pile of boxes is said to be *regular* if that pile contains equal numbers of cards with each label. Show that there exists some N such that, if $n \geq N$, then the boxes can be divided into two non-empty regular piles.

SOLUTION

Suppose a pile of boxes contains a_i copies of card i. We label the pile with the tuple $(d_2, d_3, \ldots)$ where $d_i = a_i - a_1$. So a pile is regular if and only if its label is $(0, 0, \ldots, 0)$.

It is enough to construct one regular pile, since the remaining boxes form another regular pile.

Suppose we have enough cards to ensure that there are P non-empty boxes, where P is some large number to be chosen later. We may view each of these boxes as pile. This is our first collection of piles.

Their labels all have the property that for all i, $|d_i| \leq 2022$.

We also have $\sum d_2 = 0$ where the sum is taken over all the piles.

Now suppose that the maximum value of $|d_2| = M$. We aim to form a new collection of piles such that each new pile is either one of the old piles, or is formed by combining exactly two old piles. If we have some old piles with $d_2 = M$ and others with $d_2 = -M$ we pair these up to form new piles with $d_2 = 0$. Once we have done this as many times as possible, the remaining piles with $|d_2| = M$ all have d_2 with the same sign. Consider such a pile: if it has $d_2 = M$ we combine it with any old pile with a negative value of d_2. There are sure to be enough of these, since the d_2 values sum to zero. The case where the signs are reversed is identical.

After this process we have at least $P/2$ piles. For these piles the maximum value of $|d_2|$ has decreased (by at least one) and the maximum value of $|d_i|$ for each other i has at most doubled.

Thus if we repeat this process (up to) 2022 times we will reach a situation where we have at least $P/(2^{2022})$ piles and each pile will have $d_2 = 0$ and $|d_i| \leq 2022 \times 2^{2022}$ for all other i.

Now we may run this argument again working with d_3 instead of d_2, then again with d_4 and so on. More formally, we proceed by induction.

Suppose that for some k we have a collection of P_k piles such that:

- For each pile $d_2 = d_3 = \cdots = d_k = 0$ and
- For all piles and all $i > k$ we have $|d_i| \leq M_k$ for some fixed M_k

Then, by combining the piles as described above, we can reach a situation where we have at least $P_k/(2^{M_k})$ piles, each of which has $d_{k+1} = 0$ and $|d_i| \leq 2^{M_k}$ for all i.

Setting $P_{k+1} = P_k/(2^{M_k})$ and $M_{k+1} = 2^{M_k}$ we have the same situation as before but with $k+1$ in place of k.

Thus, if we take P large enough, we can ensure that $P_{50} \geq 2$ which is enough to solve the problem.

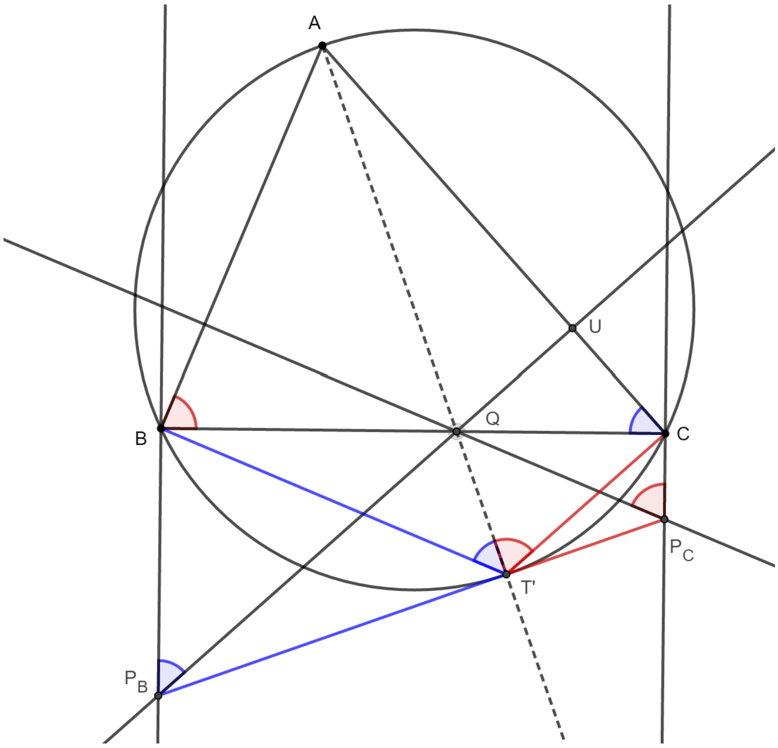
4. Let ABC be an acute angled triangle with circumcircle Γ. Let l_B and l_C be the lines perpendicular to BC which pass through B and C respectively. A point T lies on the minor arc BC. The tangent to Γ at T meets l_B and l_C at P_B and P_C respectively. The line through P_B perpendicular to AC and the line through P_C perpendicular to AB meet at a point Q. Given that Q lies on BC, prove that the line AT passes through Q.

(A minor arc of a circle is the shorter of the two arcs with given endpoints.)

SOLUTION

Note that Q is sufficient information to construct P_B and P_C.

Let T' be the second intersection of the line AQ and Γ.



Denote the foot of the perpendicular from P_B to AC by U. Then P_BBUC is cyclic, as is $ABT'C$.

Consequently: $\angle QP_BB = \angle UP_BB = \angle C = \angle AT'B$ so P_BBQT' is also cyclic.

In particular, $\angle AT'P_B = 90°$.

But the same holds for $\angle P_CT'A = 90°$.

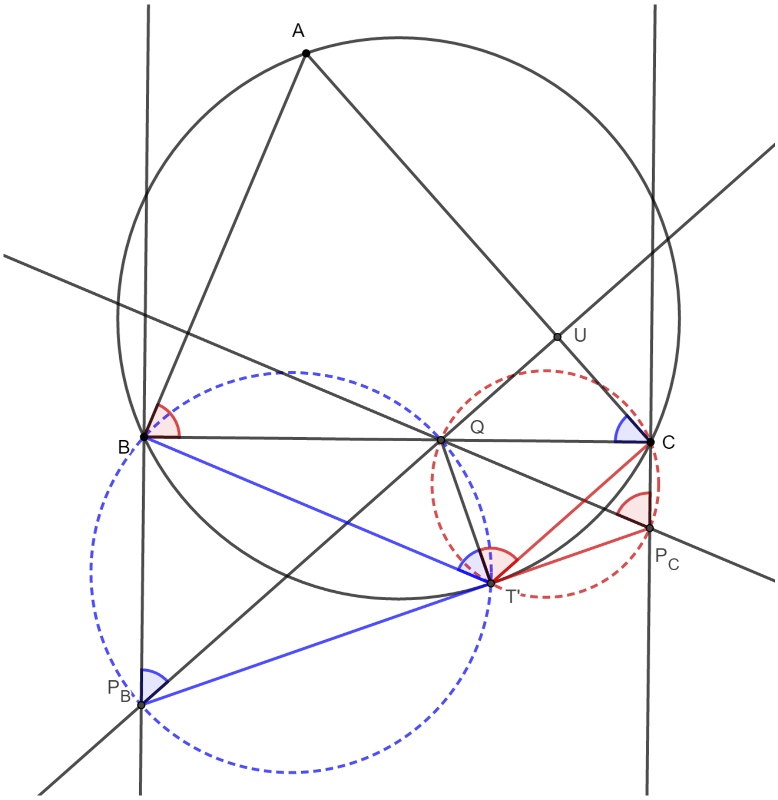
So P_B, T', P_C are collinear. This implies that $T' = T$ since the conditions in the question mean there is only one point on both the line P_BP_C and the circle Γ.

REMARK

All successful synthetic solutions to this problem began by defining a new point T' with some useful additional properties, and then proving $T' = T$. In the solution above T' is on the line AQ and on Γ. A variation on this theme is to define T' to be on the line AQ and the line P_BP_C. The solutions below provide two further alternatives.

ALTERNATIVE

Let T' be the second intersection of circles $P_C CQ$ and $P_B BQ$.



The right angles in the question show that $CUBP_B$ is cyclic (with diameter CP_B) so $\angle UP_BB = \angle C = QP_BB$.

Similarly $\angle CP_CQ = \angle B$.

The right angles show that T' is on the line P_BP_C.

The two angle facts established above show that $CT'B = \angle B + \angle C$, so T' lies on the arc BC of circle Γ. Thus $T' = T$.

Now $\angle CTA = \angle B$ using the cyclic quad $ABTC$, while $\angle CTQ = \angle CP_CQ$ using the cyclic quad $CQTP_C$.

We have already shown $\angle CP_CQ = \angle B$ so we are done.

ALTERNATIVE

Let T' be the point on Γ diametrically opposite A.

Let $\infty_{\perp\ell}$ denote the point at infinity on the line perpendicular to ℓ.

By (the converse of) angles in a semi circle, $T' = B\infty_{\perp AB} \cap C\infty_{\perp AC}$.

Applying Pappus' theorem to lines BQC and $\infty_{\perp AC}\infty_{\perp BC}\infty_{\perp AB}$ gives us that P_BP_CT' are collinear. Thus $T \equiv T'$.

If $\tilde{Q} = AT' \cap BC$ then from $BP_BA'\tilde{Q}$, $CP_CA'\tilde{Q}$ cyclic we get $P_B\tilde{Q} \perp AC$ and $P_C\tilde{Q} \perp AB$ so in fact $Q \equiv \tilde{Q}$ and thus A, Q, T are collinear on the diameter passing through A.